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# FIXED POINTS OF POINTWISE ALMOST PERIODIC HOMEOMORPHISMS ON THE TWO-SPHERE

BY

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ABSTRACT. A homeomorphism $f$ of the two-sphere $S^2$ onto itself is defined to be pointwise almost periodic (p.a.p.) if the collection of orbit closures forms a decomposition of $S^2$. It is shown that if $f: S^2 \to S^2$ is p.a.p. and orientation-reversing then the set of fixed points of $f$ is either empty or a simple closed curve; if $f: S^2 \to S^2$ is p.a.p. orientation-preserving and has a finite number of fixed points, then $f$ is shown to have exactly two fixed points.

1. **Introduction.** Every periodic mapping $f$ of the two-sphere $S^2$ to itself is topologically equivalent either to the identity, to a rotation, a reflection, or to a rotation followed by a reflection ([5] and [9]). Thus, the set of fixed points of $f$ is either empty or an $i$-sphere, $0 \leq i \leq 2$. If $f$ satisfies the weaker condition of being almost periodic (equivalent to having equicontinuous iterates) or the still weaker condition of being weakly almost periodic (the collection of orbit closures forms a continuous decomposition of $S^2$), the fixed point set is again either empty or an $i$-sphere, $0 \leq i \leq 2$ ([11] and [12]).

In this paper we study the fixed point sets of pointwise almost periodic (p.a.p.) homeomorphisms on $S^2$ (the collection of orbit closures forms a decomposition of $S^2$). In the orientation-reversing case the set of fixed points must still be either empty or a 1-sphere (Theorem 6). In the orientation-preserving case, on the other hand, there may be a continuum of fixed points together with any finite or countable number of isolated fixed points (see §6). However, if there are only a finite number of fixed points in the orientation-preserving case, then there must be exactly two (Theorem 7).

The main theorems of this paper are contained in §§5 and 6. §3 gives a summary of the results in the theory of prime ends which we use to prove the main lemma in §4.

2. **Definitions and notation.** If $A$ is a subset of a space $X$, $\mathrm{Cl}(A)$ and $\mathrm{Bd}(A)$ denote, respectively, the closure and boundary of $A$. $A$ is *nondegenerate* if $A$ is not a single point. If $f: X \longrightarrow X$ is a map, then $f|A$ denotes the restriction

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of $f$ to $A$. Homeomorphisms will always be onto.

A *domain* is a connected open set. If $U$ is a domain in $S^2$ and $M$ is a subset of a component $R$ of $S^2 - \mathrm{Cl}(U)$, then $\mathrm{Bd}(R)$ is the *outer boundary of* $U$ *with respect to* $M$.

If $f: X \longrightarrow X$ is a homeomorphism and $A \subset X$, then orbit$(A)$, the *orbit of* $A$, is the union of the set of iterates $f^n(A)$, $n = 0, \pm 1, \pm 2, \cdots$, $(f^0 = \mathrm{Id})$. The set of *period two points* of $f$ is the set $\{x \in X: f^2(x) = x\}$. Thus, the set of period two points includes the fixed points of $f$.

$f: X \longrightarrow X$ is a *recurrent homeomorphism* if, given any $x \in X$ and any neighborhood $U$ of $x$, there is a positive integer $n$ and a negative integer $m$ such that $\{f^n(x), f^m(x)\} \subset U$ [8, 10.18, p. 83].

$f: X \longrightarrow X$ is a *pointwise almost periodic* (p.a.p.) homeomorphisms if given any $x \in X$ and any neighborhood $U$ of $x$, there is a finite set $K$ of integers such that the orbit of $x$ is contained in the union of the sets $f^n(U)$, $n \in K$ [8, 4.02, p. 31]. If $X$ is a locally compact $T^2$ space, an equivalent definition is: $f: X \longrightarrow X$ is p.a.p. if the collection $\{\mathrm{Cl}(\mathrm{orbit}\ (x)): x \in X\}$ forms a decomposition of $X$ [8, 4.10, p. 32]. Note that if a homeomorphism is p.a.p. then it is recurrent.

If $U$ is a domain in $S^2$ with a nondegenerate boundary, then a *crosscut* of $U$ is an open arc in $U$ whose closure is an arc which intersects $\mathrm{Bd}(U)$ in two points. An *endcut* of $U$ is a half-open arc in $U$ whose closure is an arc which intersects $\mathrm{Bd}(U)$ in one point. If $A$ is a crosscut or an endcut of $U$, then a *subendcut* of $A$ is an endcut of $U$ which is contained in $A$.

3. **Prime ends.** This section contains the results from the theory of prime ends which we use in the next two sections. See also [12, §3] or [15].

Let $U$ be a simply connected domain in $S^2$ with a nondegenerate boundary. A *C-transformation* of $U$ onto the open unit disk $D$ is a homeomorphism $T: U \longrightarrow D$ such that the image of any crosscut of $U$ is a crosscut of $D$, and the endpoints of such images of crosscuts of $U$ are dense in the unit circle. $C$-transformations always exist [15, Appendix 2].

A collection of crosscuts $Q_1, Q_2, \cdots$, of the simply-connected domain $U$ is a *chain* if (a) the arcs $\mathrm{Cl}(Q_1), \mathrm{Cl}(Q_2), \cdots$, are pairwise disjoint; (b) $Q_n$ separates $Q_{n-1}$ from $Q_{n+1}$ in $U$; (c) there is a point on $\mathrm{Bd}(U)$ whose greatest distance from $Q_n$ approaches 0 as $n \longrightarrow \infty$. Corresponding to each $Q_n$ there is a domain $U_n$ of $U - Q_n$ containing $Q_{n+1}$. Note $U_1 \supset U_2 \supset \cdots$.

If $\{Q_i\}$, $\{R_i\}$ are chains of crosscuts and $\{U_i\}$, $\{H_i\}$ are their respective corresponding domains, then $\{Q_i\}$, $\{R_i\}$ are *equivalent chains* if for every $n$ there is an $m$ such that $H_m \subset U_n$ and $U_m \subset H_n$. Equivalent chains are said to define the same *prime end.* Thus, a prime end of $U$ is an equivalence class of chains of $U$.

If $Q_1, Q_2, \cdots$ is a chain of crosscuts in $U$, then their images $T(Q_1)$, $T(Q_2), \cdots$ under the $C$-transformation $T: U \longrightarrow D$ is a chain in $D$. If $\{Q_i\}$, $\{R_i\}$ are equivalent chains in $U$, then $\{T(Q_i)\}$, $\{T(R_i)\}$ are equivalent chains in $D$ and converge to the same point on the boundary of $D$. Thus, $T$ sets up a 1-1 correspondence between prime ends of $U$ and points on the unit circle.

If $Q_1, Q_2, \cdots$ is a chain defining the prime end $E$ and $U_1, U_2, \cdots$ are the corresponding domains of the chain, then the *impression* of $E$ is the set $\bigcap_{i=1}^{\infty} \mathrm{Cl}(U_i)$. The impression of $E$ is easily seen to be independent of which defining chain is used, Note that Impression $(E) \subset \mathrm{Bd}(U)$ and that distinct prime ends may have identical impressions.

Suppose $U$ is a simply connected domain, $T: U \longrightarrow D$ is a $C$-transformation, $E$ is a prime end of $U$, and $e$ is the point on the unit circle corresponding to $E$ under $T$. A half-open arc $B$ in $U$ *defines the prime end* $E$ if $T(B)$ is an endcut in $D$ with $e$ as a limit point. Among the half-open arcs defining the prime end $E$ there is one $A$ such that $\mathrm{Cl}(A) - A$ is minimal (contained in $\mathrm{Cl}(B) - B$ for every half-open arc $B$ defining $E$). This minimal set is the set of *principal points* of $E$ (for an alternate definition of principal point see [12, §3]). Note that a prime end defined by an endcut has just one principal point.

Given a homeomorphism $f$: $\mathrm{Cl}(U) \longrightarrow \mathrm{Cl}(U)$, with $f(U) = U$, and a $C$-transformation $T: U \longrightarrow D$, it follows that $TfT^{-1}: D \longrightarrow D$ is a $C$-transformation which may be extended to a homeomorphism $h$ of the closed unit disk onto itself [15, 4.10, p. 6; A1.7, p. 27]. If $E$ is a prime end of $U$, $e$ is the point on the unit circle corresponding to $E$ under $T$, and $h(e) = e$, then $E$ is a *fixed prime end* of $f$. If $G$ is another prime end of $U$, $p$ is the point on the unit circle corresponding to $G$ under $T$, and $p$ converges to $e$ under positive iterates of $h$, then we say *the prime end* $G$ *converges to the prime end* $E$ under positive iterates of $f$. These last two definitions are independent of the choice of the $C$-transformation $T$.

The reader unfamiliar with prime ends might attempt to show as an exercise that if $K$ is a pseudo-arc [1] and $E$ is a prime end of $S^2 - K$ then Impression $(E) = K$; also, there exists a prime end $E$ of $S^2 - K$ such that every point of $K$ is a principal point of $E$. Completing this exercise, however, is not necessary for understanding the rest of the paper.

4. **The Main lemma.** *Preliminary remark*: In the proof of Lemma 1 below, various crosscuts and endcuts are constructed in a domain $U$. Our diagrams, however, will always show the images in the open unit disk of these crosscuts and endcuts under a $C$-transformation. To avoid clumsy notation the crosscuts and endcuts will be denoted with the same letters as their $C$-images in the diagrams.

LEMMA 1. *Suppose* $f: S^2 \longrightarrow S^2$ *is an orientation-preserving homeomorphism and* $U$ *is an invariant, simply-connected domain with nondegenerate boundary. Suppose* $A$ *and* $B$ *are endcuts of* $U$ *such that* (1) *the prime end* $E$ *of* $U$ *which is defined by* $A$ *is fixed under* $f$, *and* (2) *the prime end* $F$ *of* $U$ *which is defined by* $B$ *is distinct from* $E$ *but converges to* $E$ *under positive iterates of* $f$. *Then* $f$ *is not recurrent on* $\mathrm{Bd}(U)$.

PROOF. Let $b$ be the point $\mathrm{Cl}(B) \cap \mathrm{Bd}(U)$.

*Case* 1. $b$ *is not in the impression of the prime end* $E$. Then let $V$ be a neighborhood (in $S^2$) of $b$ whose closure misses Impression $(E)$. Let $U_1 \supset U_2 \supset \cdots$ be a sequence of subdomains of $U$ such that Impression $(E) = \bigcap_{i=1}^{\infty} \mathrm{Cl}(U_i)$. Then for some $n$, $U_n \cap V = \varnothing$. If $f$ were recurrent at $b$, then subendcuts of infinitely many positive iterates of $B$ would be contained in $V$, and thus would miss $U_n$. Then the prime end $F$ would not converge to the prime end $E$.

*Case* 2. $f$ *is periodic at* $b$, *with least period* $n$. If $f(b) = b$, $n = 1$, let $Y_1$ be an open arc in $U$ such that $\mathrm{Cl}(Y_1)$ is a simple closed curve and $Y_1$ contains a subendcut of $B$ and a subendcut of $f(B)$. If $f^n(b) = b$, $n > 1$, let $Y_1, \cdots, Y_n$ be a set of pairwise disjoint crosscuts of $U$ such that each $Y_i$ contains a subendcut defining the same prime end as $f^{i-1}(B)$ and a subendcut defining the same prime end as $f^i(B)$, $1 \leqslant i \leqslant n$. Note that $\mathrm{Cl}(Y_n) \cap \mathrm{Cl}(Y_1) = \{b\} = \{f^n(b)\}$. In both cases, $\mathrm{Cl}(Y_1) \cup \cdots \cup \mathrm{Cl}(Y_n)$ forms a simple closed curve $J$ such that $J$ separates a subendcut of $A$ from the closure of some endcut $N$ of $U$ (see Figure 1 for a sketch of the $C$-images of $Y_1, \cdots, Y_n, A, N$). Note that $\mathrm{Bd}(U) \cap \mathrm{Cl}(N)$ cannot lie in Impression $(E)$.

Now, if $J'$ is the arc in the unit circle bounded by the endpoints of the $C$-images of $A$ and $B$ and containing the endpoint of the $C$-image of $f(B)$, and if $h$ is the (orientation-preserving) homeomorphism of the closed unit disk associated with $f$ (see definition of convergent prime end, §3), then $h(J') \subset J'$. The endpoint of the $C$-image of $N$ is in $J'$ and thus converges to the fixed endpoint of $J'$ under positive iterates of $h$.

But then the prime end determined by $N$ converges to the prime end $E$, and $\mathrm{Bd}(U) \cap \mathrm{Cl}(N)$ cannot lie in Impression $(E)$. Hence, by Case 1, $\mathrm{Bd}(U) \cap \mathrm{Cl}(N)$ is not a recurrent point of $f$.

*Case* 3. $b$ *is in the impression of* $E$, *but is not a periodic point.* We suppose $f$ recurrent on $\mathrm{Bd}(U)$ and derive a contradiction.

Our plan is to construct a simple closed curve $J$, made up of crosscuts of $U$ plus an arc $Y$, such that $J$ separates the endcut $A$ and some point of Impression $(E)$, (to construct $J$ we may have to modify $f$ on some subdisks of $U$), then to obtain a certain subcontinuum $L$ of Impression $(E)$ such that $L \cap Y \neq \varnothing$ but $L$ misses one component of $S^2 - J$, and finally to obtain

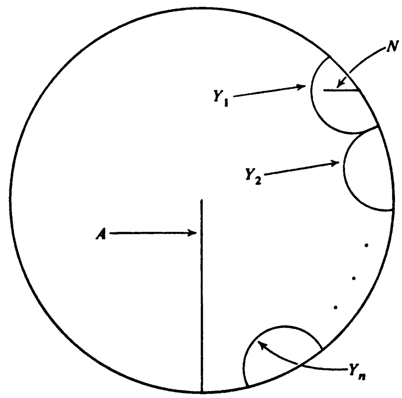


FIGURE 1

the contradiction that $f(L) = L$ but $f(L)$ must intersect both components of $S^2 - J$. We now proceed with this plan.

Since $E$ is a fixed prime end we may assume $f = \mathrm{Id}$ on $A$. For, there is an open disk $Z \subset U$ and a subendcut $A'$ of $A$ such that $\mathrm{Cl}(Z) \cap \mathrm{Bd}(U) = \mathrm{Cl}(A) \cap \mathrm{Bd}(U)$, and $A' \cup f(A') \subset Z$. Then replace $A$ by $A'$ and $f$ by $f$ followed by a homeomorphism which is the identity outside $Z$ and which is equal to $f^{-1}$ on $f(A')$.

Choose a crosscut $Q$ of $U - A$ such that (a) $Q$ has one endpoint on $A$ and the other on $\mathrm{Bd}(U)$, (b) there is a positive integer $n$ such that $Q$ separates $f^{-1}(B) \cup B \cup f(B)$ from some subendcut of $f^n(B)$ in $U - A$, (c) $\mathrm{Cl}(Q)$ is disjoint from $\mathrm{Cl}(f^{-1}(B) \cup B \cup f(B))$ and from $f^n(b)$. The existence of $Q$ follows from the fact that the prime end defined by $B$ converges to the prime end defined by $A$.

Next, choose a crosscut $X$ in $U$ such that (d) the endpoints of $X$ are $b$ and $f(b)$, (e) $X$ contains a subendcut defining the prime end, $F$, (f) the crosscuts $f^{-1}(X), X, f(X), \cdots, f^n(X)$ form a pairwise disjoint collection (here, $n$ is the integer mentioned in (b) of the preceding paragraph), and (g) $Q$ separates $f^{-1}(X)$ from $f^n(X)$ in $U - A$. See Figure 2 for a sketch of $C$-images.

The existence of $X$ follows from the facts that $f(b) \neq b$ and that $F$ converges to $E$.

Next, choose an open (in $S^2$) neighborhood $O$ of $b$ such that
$\mathrm{Cl}(O)$ is a 2-cell;
$\mathrm{Cl}(O) \cap f(\mathrm{Cl}(O)) = \varnothing$;
$[A \cup \mathrm{Cl}[(Q)] \cap [\mathrm{Cl}(O) \cup f(\mathrm{Cl}(O))] = \varnothing$;
$\mathrm{Cl}(O) \cap [\mathrm{Cl}(X) \cup f^{-1}(\mathrm{Cl}(X))]$ is an arc; and
$\mathrm{Cl}(O) \cap f^i(X) = \varnothing$ for $i = 1, 2, \cdots, n$.

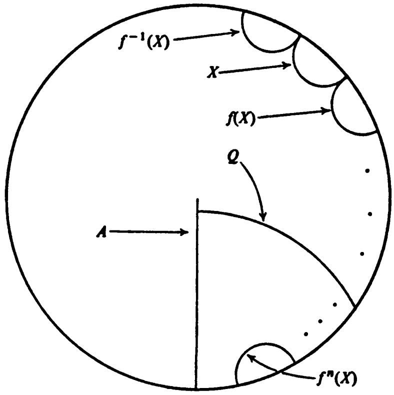


FIGURE 2

*Claim.* There is a homeomorphism $k: S^2 \longrightarrow S^2$ such that:

(1) $k = f$ on some neighborhood of $\mathrm{Bd}(U) \cup \mathrm{Cl}(O) \cup A$,

(2) $k^i(X) = f^i(X)$ for $i = -1, 0, 1, \cdots, n$, (3) for some integer $m > n$, $k^{-1}(X), X, k(X), \cdots, k^m(X)$ is a pairwise disjoint collection, $k^i(X) \cap \mathrm{Cl}(O) = \varnothing$, $n \leqslant i < m$, $k^m(X) \cap \mathrm{Cl}(O) \neq \varnothing$, and $Q$ separates $k^{-1}(X)$ and $k^i(X)$ in $U - A$, $n \leqslant i \leqslant m$.

*Proof of claim.* We construct $k$ by modifying $f$ on various subdisks of $U$.

Suppose $f^{n+1}(X) \cap Q \neq \varnothing$. Let $W$ be the component of $U - (f^n(X) \cup f^{-1}(Q))$ such that $\mathrm{Bd}(W)$ contains two disjoint subendcuts of $f^n(X)$. Let $N$ be a crosscut of $W$ such that $\mathrm{Cl}(N) \cap \mathrm{Bd}(U) = \varnothing$, the endpoints of $N$ separate $f^n(b)$ and $f^{n+1}(b)$ from $f^{-1}(Q) \cap f^n(X)$ in $\mathrm{Cl}(f^n(X))$, and $N$ is contained in a small neighborhood of $f^n(X) \cup f^{-1}(Q)$. Since $f^n(X) \cap \mathrm{Cl}(O) = \varnothing$ and $\mathrm{Cl}(Q) \cap f(\mathrm{Cl}(O)) = \varnothing$, we may choose $N$ so that $\mathrm{Cl}(N) \cap \mathrm{Cl}(O) = \varnothing$, (see Figure 3 for $C$-images).

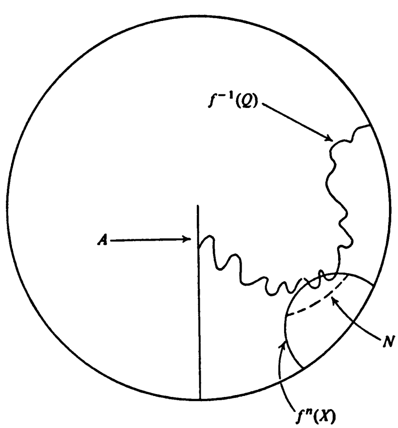


FIGURE 3

Let $Z \subset U$ be the disk bounded by $N$ and the subarc of $f^n(X)$ cut off by the endpoints of $N$. Then $Z$ misses $\mathrm{Cl}(O)$ since $\mathrm{Bd}(Z)$ misses $\mathrm{Cl}(O)$. Let $Z_1$ be a small neighborhood of $Z$ such that $\mathrm{Cl}(Z_1)$ is a subdisk of $U$, $\mathrm{Cl}(Z_1) \cap \mathrm{Cl}(O) = \emptyset$, $\mathrm{Cl}(Z_1) \cap A = \emptyset$, and $Z_1 \cap f^i(X) = \emptyset$, $i = -1, 0, 1, \cdots, n-1$. Let $g: S^2 \to S^2$ be a homeomorphism such that $g = \mathrm{Id}$ outside $Z_1$, $g = \mathrm{Id}$ on $f^n(X) - Z$, and $g(\mathrm{Bd}(Z) - N) = \mathrm{Cl}(N)$. Finally, let $h = fg$.

Note that (1) $h = f$ on some neighborhood of $\mathrm{Bd}(U) \cup \mathrm{Cl}(O) \cup A$, (2) $h^{n+1}(X) \cap Q = \emptyset$, and (3) $\{h^{-1}(X), X, h(X), \cdots, h^{n+1}(X)\} = \{f^{-1}(X), X, f(X), \cdots, f^n(X)\} \cup \{h^{n+1}(X)\}$ is a pairwise disjoint collection. (For, $h^{n+1}(X)$ can intersect the preceding images of $X$ only in $f^{-1}(X)$; otherwise, the preceding images would not be disjoint. But $Q$ separates $h^{n+1}(X)$ and $f^{-1}(X)$.)

If $\mathrm{Cl}(h^{n+1}(X)) \cap \mathrm{Cl}(O) \neq \emptyset$, then $h$ is the homeomorphism we seek. If $\mathrm{Cl}(h^{n+1}(X)) \cap \mathrm{Cl}(O) = \emptyset$, we repeat the above process, modifying $h$ to add another image of $X$ to our collection. This process must finally terminate, however, because we are assuming that $f$ is recurrent at $b$, and $b \in O$, so eventually some image of $X$ will intersect $\mathrm{Cl}(O)$. This completes the proof of our claim.

To simplify notation let us assume that $f$ requires no modification, that $f^{n+1}(X) \cap Q = \emptyset$ and $\mathrm{Cl}(f^{n+1}(X)) \cap \mathrm{Cl}(O) \neq \emptyset$.

Let $P$ be the arc $[\mathrm{Cl}(X) \cup f^{-1}(\mathrm{Cl}(X))] \cap \mathrm{Cl}(O)$. We may assume that $O$ was chosen small enough so that one of the components $V$ of $\mathrm{Cl}(O) - P$ is contained in $U - (A \cup Q)$. Then we must have $\mathrm{Cl}(f^{n+1}(X)) \cap \mathrm{Cl}(O)$ contained in $\mathrm{Cl}(O) - V$. Let $Y$ be an arc in $\mathrm{Cl}(O) - V$ from $b$ $(= \mathrm{Cl}(X) \cap f^{-1}(\mathrm{Cl}(X)))$ to a point of $\mathrm{Cl}(f^{n+1}(X))$ such that, except for its endpoints, $Y$ misses $\mathrm{Cl}(f^{-1}(X) \cup X \cup f(X) \cup \cdots \cup f^{n+1}(X))$. Then from the set $Y \cup \mathrm{Cl}(X \cup f(X) \cup \cdots \cup f^{n+1}(X))$ we may form a simple closed curve $J$.

Note that $f(Y - \{b\})$ does not intersect $J$, because $Y \subset \mathrm{Cl}(O)$ and $f(\mathrm{Cl}(O)) \cap \mathrm{Cl}(O) = \emptyset$ (this is the reason for never modifying $f$ on $\mathrm{Cl}(O)$).

Also, note that $\mathrm{Cl}(A)$ and $f(Y - \{b\})$ are in different components of $S^2 - J$. For, we may obtain an arc $C$ from $A$ to $f(b)$, which does not intersect $J - \{f(b)\}$, by starting at $A$ and traveling along close to $Q$ and then close to $f^{n-1}(X) \cup f^{n-2}(X) \cup \cdots \cup f(X)$ until we hit $f(V)$, and then traveling through $f(V)$ up to $f(b)$. Then $C \cup f(Y)$ is an arc which intersects $J$ in the piercing point $f(b)$, and thus its endpoints must lie in different components of $S^2 - J$.

Let $H$ be the component of $S^2 - J$ containing the endcut $A$. We want to construct a subcontinuum $L$ of $\mathrm{Bd}(U)$ such that $L \subset \mathrm{Cl}(H)$, $L \cap Y \neq \emptyset$, and $f(L) = L$. This will yield a contradiction.

Choose a chain $R_1, R_2, \cdots$ of crosscuts of $U$ defining $E$, with corresponding domains $U_1 \supset U_2 \supset \cdots$, such that

$$U_1 \cap (f^{-1}(X) \cup X \cup \cdots \cup f^{n+1}(X)) = \emptyset \text{ and } U_1 \cap f(V) = \emptyset.$$

Note that $f(b) \in \mathrm{Cl}(U_i)$, $i = 1, 2, \cdots$, since $b \in$ Impression $(E)$, and $E$ is a fixed prime end. Let $A_1$ be an arc in $U_1$ from $A$ to a point very close to $f(b)$. Since $U_1 \cap f(V) = \emptyset$, we must have $A_1 \cap J \neq \emptyset$, hence $A_1 \cap Y \neq \emptyset$. Let $B_1$ be the subarc of $A_1$ from $A \cap A_1$ to the first point at which $A_1$ intersects $Y$. Then $B_1 \subset \mathrm{Cl}(H)$.

Repeating this procedure, we may construct a sequence $B_1, B_2, \cdots$ of disjoint arcs of $U$ such that:

(1) $B_i$ intersects $A$ and $Y$,

(2) $B_i \subset \mathrm{Cl}(H)$,

(3) $B_i \subset U_{m(i)}$, for some $m(i) \geqslant i$.

We may assume all $B_i$'s lie on the "same side" of $A$ ($A \cup Q$ does not separate any $B_i - A$ from any $B_j - A$ in $U$) and that the $B_i$'s converge to a subcontinuum $L$ of $\mathrm{Bd}(U)$. (See Figure 4 for $C$-images.)

But then we may choose a half-open arc $T \subset U$ in a small neighborhood of $A \cup B_1 \cup B_2 \cup \cdots$ such that $T \cap A = \emptyset$, $T$ also defines the prime end $E$, and $\mathrm{Cl}(T) - T = L$. Since $f$ is orientation-preserving, both $T$ and $f(T)$ lie on the same side of $A$. Thus, by [15, 2.2, p. 2] or [14, 3.38, p. 321], either $L \subset f(L)$ or $f(L) \subset L$. But we are assuming that $f$ is recurrent on $\mathrm{Bd}(U)$ so we must have $f(L) = L$ [16, 4.12, p. 247].

Note also that $L \subset \mathrm{Cl}(H)$ and $L \cap Y \neq \emptyset$.

This easily yields a contradiction, for if $L \cap (Y - \{b\}) \neq \emptyset$, then, since $f(Y - \{b\}) \subset S^2 - \mathrm{Cl}(H)$, we cannot have $f(L) = L$; and, if $b \in L$, then $f(L) = L$ is a continuum containing $f(b)$, but every nondegenerate subcontinuum of $\mathrm{Bd}(U)$ containing $f(b)$ must intersect $S^2 - \mathrm{Cl}(H)$, since $f(V) \cap \mathrm{Bd}(U) = \emptyset$.

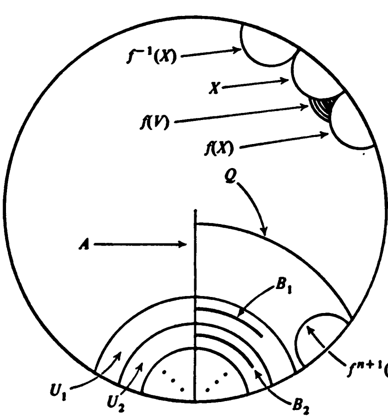


FIGURE 4

This final contradiction establishes Lemma 1.

## 5. Fixed points of orientation-reversing p.a.p. homeomorphisms.

LEMMA 2. *Let* $f\colon X \longrightarrow X$ *be a p.a.p. homeomorphism of the complete metric space* $X$, $K$ *a compact invariant subset of* $X$, *and* $U$ *an open subset of* $X$ *such that* $U - K \neq \varnothing$. *Then there is an open subset* $V$ *of* $U$ *such that the orbit of* $V$ *misses some neighborhood of* $K$.

PROOF. For each positive integer $n$, let $R(n) = \{x \in U - K\colon$ for some integer $m$, $\mathrm{dist}(f^m(x), K) < 1/n\}$. $U - K$ is complete and each $R(n)$ is open in $U - K$. Therefore, some $R(n)$ is not dense in $U - K$; otherwise, $\bigcap_{n=1}^{\infty} R(n)$ is not empty by the Baire category theorem, but $f$ cannot be p.a.p. at points of $\bigcap_{n=1}^{\infty} R(n)$. Hence, for some positive integer $m$, there is an open set $V$ in $U - K$ which misses $R(m)$. $V$ is the required open set, and the proof is complete.

LEMMA 3. *Suppose* $f\colon S^2 \longrightarrow S^2$ *is an orientation-reversing p.a.p. homeomorphism,* $p$ *is a fixed point of* $f$, *and* $Y$ *is the component of the set of period two points such that* $p \in Y$. *Then* $Y$ *is nondegenerate.*

PROOF. Assume $Y = \{p\}$. We shall establish a contradiction.

Denote by $P(2, f)$ the set of period two points of $f$.

*Claim* 1. There is an orientation-reversing, p.a.p. self-homeomorphism of $S^2$ whose set of period two points is totally disconnected and contains a fixed point.

*Proof of Claim* 1. Let $G$ be the decomposition of $S^2$ whose elements are the points of $S^2 - P(2, f)$ and the components of $P(2, f)$. $G$ is upper semicontinuous [7, p. 137], hence the decomposition space $S^2/G$ is a cactoid [16, (2.2)′, p. 172]. It is easily seen that the induced map $g = \pi f \pi^{-1}\colon S^2/G \longrightarrow S^2/G$ (where $\pi\colon S^2 \longrightarrow S^2/G$ is the decomposition map) is a p.a.p. homeomorphism, and $P(2, g) = \pi(P(2, f))$. Thus, $P(2, g)$ is totally disconnected. Now, $\pi(p)$ is not a cut point of $S^2/G$, since $\pi^{-1}(\pi(p)) = p$, so $\pi(p)$ is either an endpoint of $S^2/G$ or is contained in a true cyclic element of $S^2/G$, [16, p. 66]. If $M$ is a true cyclic element (2-sphere) containing $\pi(p)$, then $g(\pi(p)) = \pi(p)$, so $g(M) = M$, and $g|M$ is the required homeomorphism (it is clear that $g$ must be orientation-reversing).

If $\pi(p)$ is an endpoint, then there is a cutpoint $\pi(q)$ fixed by $g$ [16, 4.22, p. 247]. If $C(\pi(q), \pi(p))$ is the cyclic chain [16, p. 71] of $S^2/G$ from $\pi(q)$ to $\pi(p)$, then every cyclic element of $C(\pi(q), \pi(p))$ is invariant under $g$ [16, 4.3, p. 248]. Let $M$ be any true cyclic element of $C(\pi(q), \pi(p))$. Then $M$ is a 2-sphere and $g(M) = M$. $M$ contains a point which separates $\pi(q)$ and $\pi(p)$ in $S^2/G$ [16, 5.2, p. 71], and this point must be a fixed point of $g$ [16, 4.21, p. 247]. Thus, $g|M$ is the required orientation-reversing homeomorphism.

Claim 1 has been established.

By Claim 1 we amy assume without loss of generality that $P(2, f)$ is totally disconnected.

Let $x, y$ be points of $S^2 - \{p\}$ such that $f(x) = y$, and $f(y) = x$ ($x = y$ is allowed). The existence of $x, y$ follows from the fact that if $f^2|S^2 - \{p\}$ were fixed point free, then $S^2 - \{p\}$ would contain a point converging to $p$ under positive iterates of $f^2$ [2, Theorem 8, p. 45] (or see Theorem 7 of the present paper). Let $K$ be a continuum in $S^2$ which is invariant under $f$, which contains $\{p, x, y\}$ and which is minimal with respect to containing $\{p, x, y\}$ and being closed, connected, and invariant. By Lemma 2, there is a set $U_1 \subset K$, open in $K$, such that the orbit of $U_1$ misses a neighborhood of $P(2, f) \cap K$. Let $A_1$ be the component of $K - \text{orbit}(U_1)$ containing $p$. Note that $A_1$ is invariant and nondegenerate. Since $K$ is minimal, $A_1$ cannot contain $x$ or $y$. Let $D_1$ be the component of $S^2 - A_1$ containing $x$.

*Claim* 2. $f(D_1) \cap D_1 = \varnothing$.

*Proof of Claim* 2. Suppose $f(D_1) \cap D_1 \neq \varnothing$. Then $D_1$ is a simply-connected, invariant domain with a nondegenerate boundary. We note that $D_1$ has a prime end which is fixed under $f$. For, let $T$ be a $C$-transformation of $D_1$ onto the open unit disk, and extend $TfT^{-1}$ to an orientation-reversing homeomorphism $h$ of the closed unit disk onto itself. Then $h$ must have two fixed points on the unit circle, and these two fixed points correspond to fixed prime ends of $f$.

But then the orientation-preserving homeomorphism $f^2$ must also have a fixed prime end in $D_1$. Hence, every prime end of $D_1$ is either fixed under $f^2$ or converges to a fixed prime end under positive iterates of $f^2$ [3, Lemma 14].

Since $P(2, f)$ is totally disconnected and closed, there is an endcut $B$ in $D_1$ such that $\text{Cl}(B) \cap \text{Bd}(D_1)$ is not in $P(2, f)$. Thus, if $F$ is the prime end defined by $B$, the principal point of $F$ is not fixed under $f^2$, and so $F$ is not a fixed prime end of $f^2$ [12, Lemma 1]. Thus, $F$ converges under positive iterates of $f^2$ to a fixed prime end $E$. Since every principal point of $E$ is fixed under $f^2$ [12, Lemma 1] and $P(2, f)$ is totally disconnected, $E$ has only one principal point, [14, Corollary, p. 275]. Thus, there is an endcut $A$ of $D_1$ which defines $E$. But then, by Lemma 1, $f^2$ is not recurrent on $\text{Bd}(D_1)$. This contradicts the fact that, since $f$ is p.a.p. on $\text{Bd}(D_1)$, $f^2$ is p.a.p. on $\text{Bd}(D_1)$ [8, p. 31].

Claim 2 is established.

Since $x \in D_1$ and $A_1$ is invariant, $f^2(D_1) = D_1$, hence $\text{Bd}(D_1) \cup f(\text{Bd}(D_1))$ is an invariant subset of $f$.

*Claim* 3. There is a nondegenerate, invariant subcontinuum $L$ of $A_1$ such that if $O$ is the component of $S^2 - L$ containing $x$, then $p \in \text{Bd}(O)$.

*Proof of Claim* 3. If $p \in \text{Bd}(D_1)$, then $\text{Bd}(D_1) \cup f(\text{Bd}(D_1))$ is the required subcontinuum. Suppose $p \notin \text{Bd}(D_1)$. Let $B_1$ be an invariant subcontinuum of

$A_1$ containing $\{p\} \cup \mathrm{Bd}(D_1) \cup f(\mathrm{Bd}(D_1))$ and minimal with respect to these properties. Let $V_2$ be a ball of radius less than ½, centered at $p$. By Lemma 2, there is a set $U_2 \subset B_1 \cap V_2$, open in $B_1$, such that the orbit of $U_2$ misses a neighborhood of $[B_1 \cap P(2,f)] \cup \mathrm{Bd}(D_1) \cup f(\mathrm{Bd}(D_1))$. Let $A_2$ be the component of $B_1 - \mathrm{orbit}(U_2)$ containing $p$. Then $A_2$ is invariant, nondegenerate; and, by the minimality of $B_1$, $A_2$ does not intersect $\mathrm{Bd}(D_1) \cup f(\mathrm{Bd}(D_1))$. If $D_2$ is the component of $S^2 - A_2$ containing $\mathrm{Bd}(D_1)$ (and thus containing $\mathrm{Cl}(D_1)$), then by the same proof as in Claim 2, $f(D_2) \cap D_2 = \varnothing$. Also, $f^2(D_2) = D_2$, since $x \in D_2$. And clearly, we must have $\mathrm{orbit}(U_2) \cap D_2 \neq \varnothing$. But then, for some integer $n$, $f^n(D_2) \cap U_2 \neq \varnothing$. Thus, either $D_2 \cap U_2 \neq \varnothing$ or $f(D_2) \cap U_2 \neq \varnothing$. Hence, $D_2 \cup f(D_2)$ intersects a neighborhood of $p$ of radius less than ½.

If $p \in \mathrm{Bd}(D_2)$, then $\mathrm{Bd}(D_2) \cup f(\mathrm{Bd}(D_2))$ is the required subcontinuum. If $p \notin \mathrm{Bd}(D_2)$, we continue the above process. Either we terminate at a finite stage or else we obtain a sequence $D_1, D_2, \cdots$ of simply-connected domains such that:

(1) $D_1 \subset \mathrm{Cl}(D_1) \subset D_2 \subset \mathrm{Cl}(D_2) \subset D_3 \subset \cdots$;

(2) $\mathrm{Bd}(D_i) \subset A_1$, $f(D_i) \cap D_i = \varnothing$, $f^2(D_i) = D_i$, for each $i$;

(3) $f(D_i) \cup D_i$ intersects a neighborhood of $p$ of radius less than $1/i$, for each $i$.

If we let $O = \bigcup_{i=1}^{\infty} D_i$, and $L = \mathrm{Bd}(O) \cup f(\mathrm{Bd}(O))$, then $L$ is the required subcontinuum, with $p \in \mathrm{Bd}(O) \cap f(\mathrm{Bd}(O))$.

Claim 3 is established.

As in Claim 2, we must have $f(O) \cap O = \varnothing$. And since $x \in O$, $f^2(O) = O$. Now let $L_1$ be the outer boundary of $O$ with respect to $f(O)$. Since $p \in \mathrm{Bd}(O) \cap f(\mathrm{Bd}(O))$, we must have $p \in L_1$. By Lemma 2, there is a set $V \subset L_1 \cup f(L_1)$, open in $L_1 \cup f(L_1)$, such that $\mathrm{orbit}(V)$ misses a neighborhood of $P(2,f) \cap [L_1 \cup f(L_1)]$. Let $X$ be the component of $[L_1 \cup f(L_1)] - \mathrm{orbit}(V)$ containing $p$. Then $X$ is nondegenerate and invariant. Let $W$ be the component of $S^2 - X$ containing $x$. As before, $f(W) \cap W = \varnothing$. Since $L_1 - \mathrm{orbit}(V)$ does not separate $x$ and $f(x)$ [13, p. 176], $\mathrm{Bd}(W)$ must contain points of $f(L_1) - L_1$. Let $q$ be a point of $\mathrm{Bd}(W) \cap f(L_1)$, $q \neq p$. Since $f(L_1) \subset \mathrm{Bd}(f(O))$, we note that $f(O) \cup \{q\} \cup W$ is a connected set containing $x$ and $f(x)$. Now let $X_1$ be an invariant subcontinuum of $X$ containing $p$ and $\mathrm{Cl}(\mathrm{orbit}(q))$ and minimal with respect to these properties. By Lemma 2, there is a set $Z \subset X_1$, open in $X_1$, such that $\mathrm{orbit}(Z)$ misses a neighborhood of $[P(2,f) \cap X_1] \cup \mathrm{Cl}(\mathrm{orbit}(q))$. Let $X_2$ be the component of $X_1 - \mathrm{orbit}(Z)$ containing $p$. Then $X_2$ is nondegenerate, invariant, and $X_2$ misses the set $f(O) \cup \{q\} \cup W$. But then the component of $S^2 - X_2$ containing $f(O) \cup \{q\} \cup W$ is invariant, and we arrive at exactly the same contradiction as in the proof of Claim 2.

This final contradiction establishes Lemma 3.

LEMMA 4. *Suppose* $f: S^2 \longrightarrow S^2$ *is a homeomorphism and* $U$ *is a simply-connected domain such that* $f(U) \cap U = \varnothing$ *and* $f = \mathrm{Id}$ *on* $\mathrm{Bd}(U)$. *Then* $\mathrm{Bd}(U)$ *is a simple closed curve.*

PROOF. Fix a point $x \in \mathrm{Bd}(U)$.

*Claim.* There is an endcut $A$ of $U$ such that $\mathrm{Cl}(A) \cap \mathrm{Bd}(U) = \{x\}$.

*Proof of Claim.* Let $R_1, R_2, \cdots$ be a chain of crosscuts defining a prime end whose impression contains $x$, and let $U_1 \supset U_2 \supset \cdots$ be the corresponding subdomains of $U$ (so $x \in \bigcap_{i=1}^{\infty} \mathrm{Cl}(U_i)$). For each $i$, let $J_i$ be the simple closed curve $\mathrm{Cl}(R_i) \cup f(\mathrm{Cl}(R_i))$, and let $D_i$ be the component of $S^2 - J_i$ containing $U_i$. Then $\mathrm{Cl}(D_1) \supset \mathrm{Cl}(D_2) \supset \cdots$. Since the diameters of the $R_i$'s converge to zero, the diameters of the $J_i$'s converge to zero, and so $\bigcap_{i=1}^{\infty} \mathrm{Cl}(U_i) \subset \bigcap_{i=1}^{\infty} \mathrm{Cl}(D_i)$ is the single point $x$. Thus, if $A$ is a half-open arc in $U$ such that each $R_i$ separates the endpoint of $A$ from some terminal portion of $A$, we see that $\mathrm{Cl}(A) - A \subset \bigcap_{i=1}^{\infty} \mathrm{Cl}(U_i) = \{x\}$. Thus, $A$ is the required endcut and the claim is proved.

By [16, p. 58], $\mathrm{Bd}(U)$ is a simple closed curve provided any two points of $\mathrm{Bd}(U)$ separate $\mathrm{Bd}(U)$. Let $x, y$ be any two points of $\mathrm{Bd}(U)$. By the claim there is a crosscut $A$ of $U$ such that $\mathrm{Cl}(A) \cap \mathrm{Bd}(U) = \{x, y\}$. Then $J = \mathrm{Cl}(A) \cup f(\mathrm{Cl}(A))$ is a simple closed curve, and $\mathrm{Bd}(U)$ must intersect both components of $S^2 - J$ (otherwise $A$ and $f(A)$ would lie in the same component of $S^2 - \mathrm{Bd}(U)$). But then $J$ separates $\mathrm{Bd}(U) - \{x, y\}$ in $S^2$, hence $\{x, y\}$ separates $\mathrm{Bd}(U)$. The proof of Lemma 4 is complete.

LEMMA 5. *Suppose* $f: S^2 \longrightarrow S^2$ *is an orientation-reversing, p.a.p. homeomorphism,* $K$ *is a component of the set of period two points of* $f$ *such that* $K$ *contains a fixed point, and* $U$ *is a component of* $S^2 - K$. *Then* $f(U) \cap U = \varnothing$, *and* $f^2(U) = U$.

PROOF. First we show that $f(U) \cap U = \varnothing$. Suppose not. Then, since $K$ is invariant, $f(U) = U$. Let $G$ be the (upper semicontinuous) decomposition of $S^2$ whose only nondegenerate element is $S^2 - U$. Then the decomposition space $S^2/G$ is homeomorphic to $S^2$ [16, (2.1)′, p. 171], and the induced map $g = \pi f \pi^{-1}: S^2/G \longrightarrow S^2/G$ (where $\pi: S^2 \longrightarrow S^2/G$ is the decomposition map) is easily seen to be a p.a.p. orientation-reversing homeomorphism. If we denote the set of period two points of $f$ by $P(2, f)$, then $\pi(P(2, f)) = P(2, g)$, and $\pi(S^2 - U)$ is a degenerate component of $P(2, g)$. This contradicts Lemma 3. Hence $f(U) \cap U = \varnothing$.

Now suppose $f^2(U) \neq U$. Then $f^2(U) \cap U = \varnothing$. And $f^2 = \mathrm{Id}$ on $\mathrm{Bd}(U)$, since $K \subset P(2, f)$. Hence, by Lemma 4, $\mathrm{Bd}(U)$ is a simple closed curve. But then $U$ must be one component of $S^2 - \mathrm{Bd}(U)$ and $f^2(U)$ must be the other component. This is impossible because $f^2$ is orientation-preserving. Hence $f^2(U) = U$, and the proof of Lemma 5 is complete.

THEOREM 6. *Suppose $f: S^2 \longrightarrow S^2$ is a p.a.p. orientation-reversing homeomorphism which has a fixed point. Then the set of fixed points of $f$ is a simple closed curve.*

PROOF. Let $K$ be a component of the set of period two points of $f$ such that $K$ contains a fixed point of $f$. Let $V_1, V_2, \cdots$ be a list of the components of $S^2 - K$. By Lemma 5, $f(V_i) \cap V_i = \varnothing$ and $f^2(V_i) = V_i$, for each $i$. Let $A_1 = V_1$ and $B_1 = f(V_1)$, and suppose we have defined sets $A_n, B_n$ which are unions of components of $S^2 - K$ such that:

(1) $A_n \cap B_n = \varnothing$;

(2) $V_1 \cup V_2 \cup \cdots \cup V_n \subset A_n \cup B_n$;

(3) for each $i$, $V_i$ intersects $A_n$ if and only if $f(V_i)$ intersects $B_n$.

Form $A_{n+1}, B_{n+1}$ as follows: if $V_{n+1}$ intersects $A_n \cup B_n$, let $A_{n+1} = A_n$ and $B_{n+1} = B_n$; if $V_{n+1}$ does not intersect $A_n \cup B_n$, let $A_{n+1} = A_n \cup V_{n+1}$, and $B_{n+1} = B_n \cup f(V_{n+1})$.

Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$. Then $A \cap B = \varnothing, f(A) = B$, $f(B) = A$, and $S^2 = A \cup B \cup K$.

Define a map $g: S^2 \longrightarrow S^2$ by

$$g(x) = \begin{cases} f(x) \ \ (= f^{-1}(x)), & \text{if } x \in K, \\ f(x), & \text{if } x \in A, \\ f^{-1}(x), & \text{if } x \in B. \end{cases}$$

It is easily checked that $g$ is a periodic, orientation-reversing homeomorphism, and that the set of fixed points of $g$ is identical with the set of fixed points of $f$. The set of fixed points of $g$ is a simple closed curve by [5]. The proof of Theorem 6 is complete.

6. **Orientation-preserving homeomorphisms.** In the orientation-preserving case, the similarity between the fixed point sets of p.a.p. homeomorphisms and the fixed point sets of periodic (or weakly almost periodic, see [12]) homeomorphisms no longer holds.

EXAMPLE. Let $D_1, D_2, \cdots$ be a (finite or infinite) collection of closed disks in $S^2$ such that the union of the $D_i$'s is compact and locally connected, and if $i \neq j$, then $D_i \cap D_j$ is the south pole $p_0 \in S^2$. For each $i$, let $g_i$ be a homeomorphism of $D_i$ onto the disk $\{(r, \theta) \in R^2: r \leqslant 1\}$ ($(r, \theta)$ polar coordinates). Define $g: S^2 \longrightarrow S^2$ by setting $g = \mathrm{Id}$ outside the union of the $D_i$'s and setting $g|D_i = g_i^{-1} f g_i$ where $f(r, \theta) = (r, \theta + 1 - r)$. Then $g$ is orientation-preserving and p.a.p., and the number of isolated fixed points of $g$ is equal to the number of disks $D_i$.

We do, however, have the following partial result.

THEOREM 7. *Suppose* $f\colon S^2 \longrightarrow S^2$ *is a recurrent, orientation-preserving homeomorphism with a finite number of fixed points. Then* $f$ *has exactly two fixed points.*

PROOF. The proof consists mostly of combining known results. Let $p$ be a fixed point of $f$, and let $U$ be a neighborhood of $p$ which contains no other fixed points.

*Claim.* The fixed point index $i(f, U)$ of $f$ on $U$ is equal to $+1$ (see [12, §4] for a short discussion of the local fixed point index $i(f, U)$ or [4] for a more comprehensive treatment).

*Proof of Claim.* We may assume $U \neq S^2$ so that $U$ may be identified with a subset of the plane $R^2$. Using the construction in the proof of [6, Lemma 1], we obtain an orientation-preserving homeomorphism $h\colon R^2 \longrightarrow R^2$ such that $p$ is the only fixed point of $h$, and $h = f$ on some neighborhood $V \subset U$, with $p \in V$. (In the proof of Lemma 1 of [6], choose the set $D_1$ of that proof to be any neighborhood of $p$ such that $\mathrm{Cl}(D_1)$ is a disk contained in $U$. The construction then easily yields the required homeomorphism $h$.) Since no point of $V$ converges to $p$ under positive or negative iterates of $h$, there is a point $x \in V - \{p\}$ whose orbit under $h$ is contained in $V$ [8, 10.28, p. 85]. If $x$ is not a period two point, the construction given in [10, p. 89] or [2, p. 45], yields an arc $A$ in $R^2$ (a so-called *translation arc*) such that $x \in A$, one of the endpoints of $A$ is the image under $h$ of the other endpoint, and $A \cap h(A)$ is this common endpoint. Since $h$ is recurrent at $x$ it is easy to see that $A$ can be chosen so that $h^n(x) \in A$ for some $n > 1$. If $x$ has period two then the construction of [10, p. 89] yields either a translation arc as in the previous sentence or an arc $A$ joining $x$ and $h(x)$ such that $A \cap h(A) = \{x, h(x)\}$. If $A \cap h(A) = \{x, h(x)\}$, let $J$ be the simple closed curve $A \cup h(A)$. In the case $A \cap h(A)$ is a single point, let $J$ be a simple closed curve constructed from $A \cup h(A) \cup \cdots \cup h^n(A)$, where $n$ is the least integer greater than one such that $A \cap h^n(A) \neq \varnothing$. Then if $D$ is the bounded component of $R^2 - J$, $D$ must contain a fixed point of $h$ [10, Lemma 1.1, p. 89 and Proof of Theorem 1, p. 90]. Thus, $p \in D$. And in fact, the fixed point index $i(h, D)$ is equal to 1, for, Lemma 1.1 of [10, p. 89], shows that as a point $t$ makes one positive circuit of $J$, the vector from $t$ to $h(t)$ turns through an angle of $2\pi$; thus, the map from $J$ to the unit circle which takes $t$ to $(h(t) - t)/|h(t) - t|$ has degree 1; thus, the fundamental class of $H_2(D, D - \{p\})$ is mapped to the fundamental class of $H_2(R^2, R^2 - \{0\})$ by the homomorphism induced by $\mathrm{Id} - h$, and thus $i(h, D) = 1$ [4]. But $D - V$, $V - D$ contain no fixed point of $h$, and $U - V$ contains no fixed point of $f$, hence:

$$1 = i(h, D) = i(h, V) = i(f, V) = i(f, U).$$

The claim is established.

Thus, if $U_1, \cdots, U_m$ is a collection of pairwise disjoint open subsets of $S^2$ whose union contains all fixed points of $f$ and such that each $U_i$, $1 \leq i \leq m$, contains exactly one fixed point of $f$, then:

$$\sum_{j=1}^{m} i(f, U_j) = L(f),$$

where $L(f)$ is the Lefschetz number of $f$ [12, §4]. But $L(f) = 2$, since $f$ is an orientation-preserving homeomorphism. By our claim, $i(f, U_j) = 1$, $1 \leq j \leq m$, hence $m = 2$, and the proof of Theorem 7 is complete.

We conclude with two questions.

*Question* 1. Is there an example of a homeomorphism $f: S^2 \longrightarrow S^2$ such that $f$ is recurrent but not p.a.p.?

*Question* 2. Suppose $f: S^2 \longrightarrow S^2$ is an orientation-preserving p.a.p. homeomorphism such that no component of the set of fixed points separates $S^2$. Must the set of fixed points have exactly two components?

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